Assignment 1 Solutions

COMP 2411, Session 1, 2004

Q1: (1)

- $(\neg\neg p \leftrightarrow (p \leftrightarrow (q \vee r)))$
- $(\neg(\neg p \leftrightarrow p) \leftrightarrow (q \vee r))$
- $((\neg(p \to q) \vee (r \vee s)) \to q)$
- $(p \leftrightarrow ((\neg p \vee q) \to (p \wedge (q \vee r))))$
- $((\neg p \vee (q \vee (r \wedge s))) \leftrightarrow (p \wedge \neg p))$

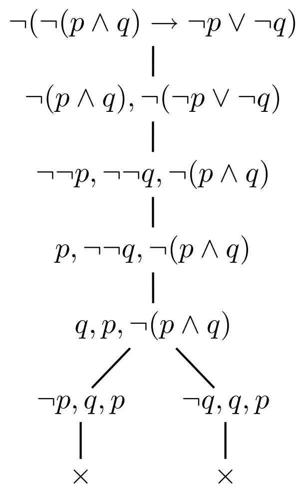
(2)

- $\neg\neg p \leftrightarrow p \leftrightarrow q \vee r$
- $\neg(\neg p \leftrightarrow p) \leftrightarrow q \vee r$
- $\neg(p \to q) \vee r \vee s \to q$
- $p \leftrightarrow \neg p \vee q \to p \wedge (q \vee r)$
- $\neg p \vee q \vee r \wedge s \leftrightarrow p \wedge \neg p$

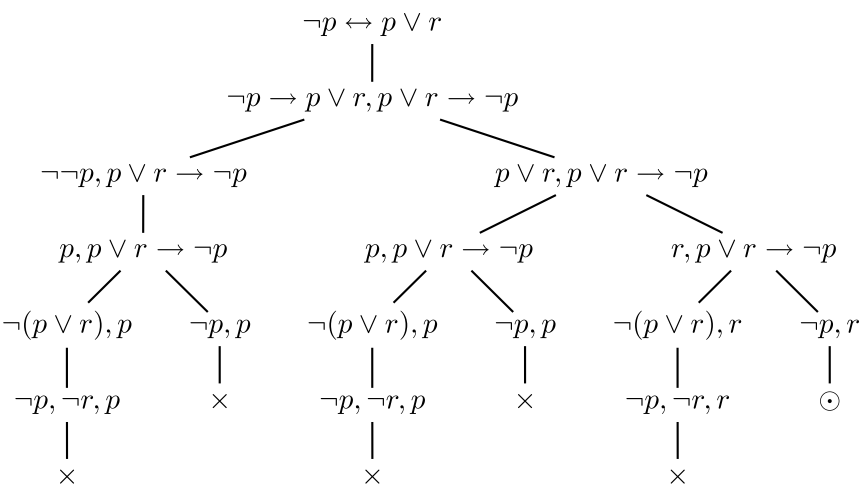
Q2; Let p be a propositional atom and let φ be a formula built from $\to$ and $\vee$ only. If all atoms in φ are given the value true then φ takes the value true. Hence $\neg p$ is not logically equivalent to φ. Hence negation is not definable from $\to$ and $\vee$ only.

Let p, q be propositional atoms and let φ be a formula built from $\neg$ and $\leftrightarrow$ only. Note that for all formulas ψ, χ, $\neg(\psi \leftrightarrow \chi)$ is logically equivalent to $\psi \leftrightarrow \neg\chi$. Hence we can without loss of generality assume that negation only applies to atomic formulas in φ. Also note that for all formulas ψ, χ, if χ is given the value true then $\psi \leftrightarrow \chi$ is logically equivalent to ψ, and if χ is given the value false then $\psi \leftrightarrow \chi$ is logically equivalent to $\neg\psi$. Hence if all atoms in φ except p and q are given the value true, then φ becomes logically equivalent to a tautology or to a contradiction or to a formula φ' built from p, q, $\leftrightarrow$ and $\neg$ only, where negation is only applied to atomic formulas. Consider the last case. Since $\leftrightarrow$ is commutative and associative, we can assume that either p does not occur in φ', or that p occurs only once in φ', or that if p occurs n times, $n > 1$, in φ' then φ' contains a subformula of the form $\overbrace{p \leftrightarrow \ldots \leftrightarrow p}^{n}$; we can assume the same for $\neg p$, q and $\neg q$. Since $p \leftrightarrow p$ is valid, if $\overbrace{p \leftrightarrow \ldots \leftrightarrow p}^{n}$ occurs in φ' for some $n > 1$ then $\overbrace{p \leftrightarrow \ldots \leftrightarrow p}^{n}$ can be reduced to a tautology if n is even, and to p if n is odd; the same observation holds for $\neg p$, q and $\neg q$. Finally, since $p \leftrightarrow \neg p$ are $q \leftrightarrow \neg q$ are unsatisfiable, φ' is logically equivalent to a tautology, or to a contradiction, or to p, or to $\neg p$, or to q, or to $\neg q$, or to $p \leftrightarrow q$, or to $\neg p \leftrightarrow q$, or to $p \leftrightarrow \neg q$, or to $\neg p \leftrightarrow \neg q$. None of these formulas is equivalent to $p \vee q$. Hence disjunction is not definable from $\neg$ and $\leftrightarrow$ only.

Q3: Since the following tableau is$_\times$ $\neg(\neg(p \wedge q) \to \neg p \vee \neg q)$ is unsatisfiable.



The second formula is true iff p is false and r is true:



Q4: Let formulas φ, ψ be such that $\varphi < \psi$. Let E be the set of propositional atoms that occur in φ or ψ. There exists a partition of E into 2 classes E^+ and E^- such that if all members of E^+ are true and all members of E^- are false then φ gets the value false whereas ψ gets the value true. Let ξ^+ be the conjunction of all members of E^+ and let ξ^- be the conjunction of the negations of the members of E^-. Take a new propositional atom x, and put $\chi = \varphi \vee (\xi^+ \wedge \xi^- \wedge x)$. Obviously, $\varphi \models \chi$. When all members of E^+ are true, all members of E^- are false and x is true, φ is false whereas $\xi^+ \wedge \xi^- \wedge x$ is true, hence χ is true, hence $\chi \not\models \varphi$. So we have shown that $\varphi < \chi$. Since $\varphi \models \psi$ and $\xi^+ \wedge \xi^- \models \psi$, we infer that $\chi \models \psi$. When all members of E^+ are true, all members of E^- are false and x is false, both φ and $\xi^+ \wedge \xi^- \wedge x$ are false whereas ψ is true, hence $\psi \not\models \chi$. So we have shown that $\chi < \psi$, and we are done.

Q5: Let a set X of formulas be given. If X has no model then it suffices to take $Y = \{p \wedge \neg p\}$, so suppose that X has a model. If $X = \emptyset$ then it suffices to take $Y = \emptyset$, so suppose $X \neq \emptyset$. Let $(\psi_i)_{i \in \mathbb{N}}$ be an enumeration of X (possibly with repetitions). We inductively define a sequence of sets $(X_i)_{i \in \mathbb{N}}$ as follows. Put $X_0 = \emptyset$. Let $i \in \mathbb{N}$ be given, and suppose that X_i has been defined. If

$X_i \models \psi_i$ then put $X_{i+1} = X_i$, otherwise put $X_{i+1} = X_i \cup \{\psi_i\}$. Define Z as $\bigcup_{i\in\mathbb{N}} X_i$. If $Z = \emptyset$ then again it suffices to take $Y = \emptyset$, so suppose $Z \neq \emptyset$. If Z is finite then it suffices to take for Y the singleton consisting of the conjunction of the members of Z, so suppose that Z is infinite. Fix an enumeration $(\varphi_i)_{i\in\mathbb{N}}$ of Z. Define Y as $\{\varphi_1, \neg\varphi_1 \vee \varphi_2, \neg\varphi_1 \vee \neg\varphi_2 \vee \varphi_3 \ldots\}$. It is immediately verified that Y has the same models as X. By construction of Z it is possible to make φ_1 false. Moreover, if φ_1 is false then all members of $Y \setminus \{\varphi_1\}$ are true, hence $Y \setminus \{\varphi_1\} \not\models \varphi_1$. Let a nonnull $i \in \mathbb{N}$ be given. By construction of Z it is possible to make $\varphi_1, \ldots, \varphi_i$ true and φ_{i+1} false. If $\varphi_1, \ldots, \varphi_i$ are true and φ_{i+1} is false then $\psi = \neg\varphi_1 \vee \ldots \vee \neg\varphi_i \vee \varphi_{i+1}$ is false whereas all members of $Y \setminus \{\psi\}$ are clearly true, hence $Y \setminus \{\psi\} \not\models \psi$. We conclude that Y is independent.