Lecture notes 13.0

First-order logic: semantics

COMP 2411, session 1, 2004

Introduction (1)

It is important to make the distinction between concrete and abstract interpretations.

Often we define the nonlogical symbols of a vocabulary having a concrete interpretation in mind. The semantics of the predicate calculus is based on abstract interpretations, *i.e.*, mathematical structures.

Whether a mathematical structure can offer an adequate picture of the real world is a question that can be addressed, but outside the realm of mathematical logic.

The semantics of the predicate calculus defines and studies the notion: a formula is true/false in a structure.

Introduction (2)

Choosing English words for function and predicate symbols usually indicates that we have an intended interpretation in mind. A formula such as

$$\forall \mathrm{x} \forall \mathrm{y}((\mathrm{mother}(\mathrm{x}) \wedge \mathrm{child\_of}(\mathrm{y}, \mathrm{x})) \rightarrow \mathrm{loves}(\mathrm{x}, \mathrm{y}))$$

gets a meaning when we 'read' it in English: every mother loves her children.

What is meant by *meaning* here

- is a relationship between an English sentence and the physical world;
- is hard to define precisely;
- is rich and not reducible to true or false.

Introduction (3)

In logic, what is meant by *meaning*

- is a relationship between a closed (first-order) formula and an abstract world, namely, an algebraic structure;
- is perfectly well defined and formalized;
- is reducible to true or false.

In logic, giving a meaning to

$$\forall \mathrm{x} \forall \mathrm{y}((\mathrm{mother}(\mathrm{x}) \wedge \mathrm{child\_of}(\mathrm{y}, \mathrm{x})) \rightarrow \mathrm{loves}(\mathrm{x}, \mathrm{y}))$$

is equivalent to giving a meaning to

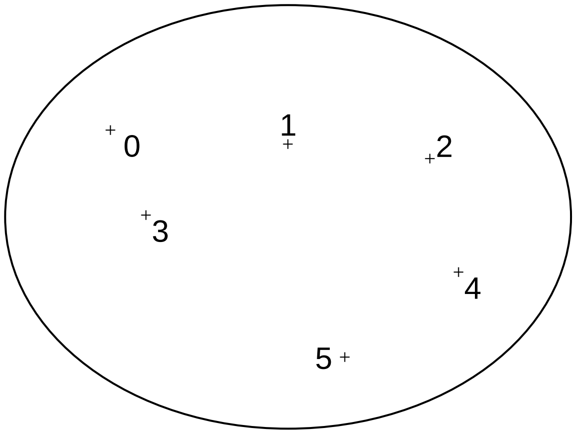
$$\forall x \forall y((P(x) \wedge Q(y, x)) \rightarrow R(x, y)).$$

An example

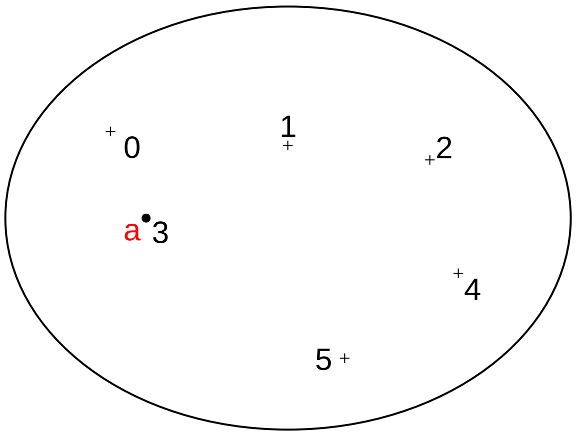
Consider a vocabulary V whose nonlogical symbols are a constant "a", a unary function symbol "s", a unary predicate symbol "P", and a binary predicate symbol "R". A structure $\mathfrak{M}$ over V is defined as:

- a nonempty X set of individuals—the domain of $\mathfrak{M}$
- an interpretation of "a" in $\mathfrak{M}$: a member of X
- an interpretation of "s" in $\mathfrak{M}$: a unary operation over X (function from X into X)
- an interpretation of P in $\mathfrak{M}$: a unary relation over X (subset of X)
- an interpretation of R in $\mathfrak{M}$: a binary relation over X (subset of X^2)

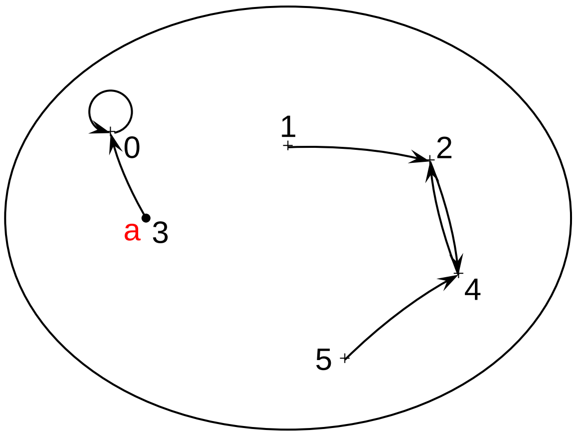
The domain of $\mathfrak{M}$...



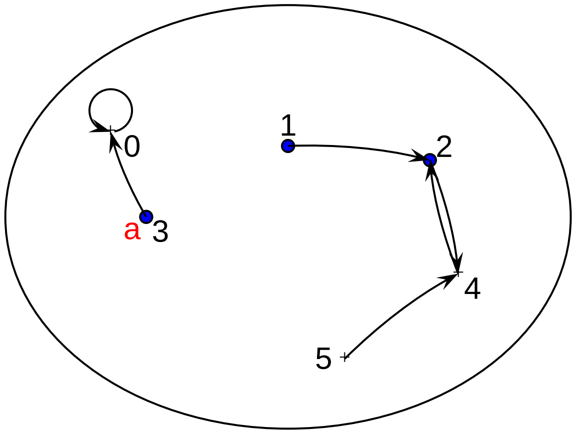
...plus the interpretation of "a"...



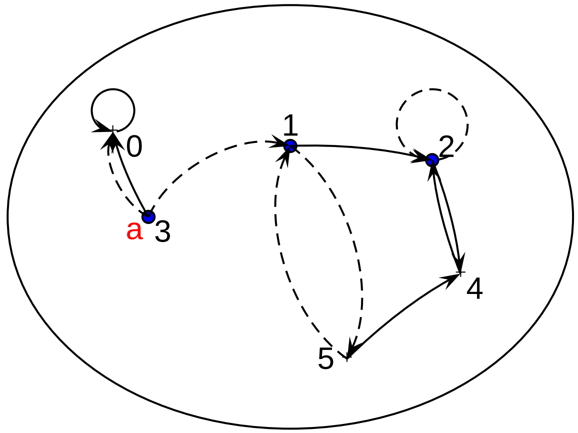
...plus the interpretation of "s"...



. . . plus the interpretation of "P". . .



. . . plus the interpretation of "R"



Who has a name?

3 has a unique name: a

0 has infinitely many names: s(a), s(s(a)), s(s(s(a)))...

1, 2, 4 and 5 have no name, though we can refer to them indirectly if we have equality in the language:

- 1 is the unique guy x such that $\exists y \exists z(y \neq z \wedge R(y,x) \wedge R(z,x))$
- 2 is the unique guy x such that $R(x,x)$
- 4 is the unique guy x such that $\exists y(R(y,y) \wedge s(y) = x)$
- 5 is the unique guy x such that $s(s(x)) \neq x \wedge R(s(s(x)), s(s(x)))$

Usually, most individuals in a structure have no name and cannot even be referred to indirectly.

Description of $\mathfrak{M}$

Because $\mathfrak{M}$ is finite, we can describe $\mathfrak{M}$, up to isomorphism, *i.e.*, up to the nature of its individuals, as the unique structure that satisfies the following formula:

Cardinality of the domain:

$$\exists x_0 \ldots \exists x_5 (x_0 \neq x_1 \wedge x_0 \neq x_2 \wedge \ldots \wedge x_4 \neq x_5 \wedge \forall x(x = x_0 \vee \ldots \vee x = x_5) \wedge \ldots$$

Description of s: $s(x_0) = x_0 \wedge s(x_1) = x_2 \wedge s(x_2) = x_4 \wedge s(x_3) = x_0 \wedge s(x_4) = x_2 \wedge s(x_5) = x_4 \wedge \ldots$

Description of P:
$\forall x(P(x) \leftrightarrow (x = x_1 \vee x = x_2 \vee x = x_3)) \wedge \ldots$

Description of R: $\forall y \forall z(R(y,z) \leftrightarrow ((y = x_3 \wedge z = x_0) \vee (y = x_3 \wedge z = x_1) \vee (y = x_1 \wedge z = x_5) \vee (y = x_5 \wedge z = x_1) \vee (y = x_2 \wedge z = x_2))))$.

Structures: definition

Given a vocabulary V, a structure $\mathfrak{M}$ over V is a nonempty set, called the domain of $\mathfrak{M}$, denoted $|\mathfrak{M}|$, together with an interpretation of all function and predicate symbols in V, *i.e.*:

- for each n-ary function symbol $f \in V$, an n-ary operation $f^{\mathfrak{M}}$ over $|\mathfrak{M}|$;
- for each n-ary predicate symbol $R \in V$, an n-ary relation $R^{\mathfrak{M}}$ over $|\mathfrak{M}|$.

In particular, each constant $c \in V$ is interpreted as a member $c^{\mathfrak{M}}$ of $|\mathfrak{M}|$.

We can represent a structure graphically if its domain is small, no function symbol is of arity greater than 1, and no predicate symbol is of arity greater than 2.

$\mathfrak{M}$ defined formally

With our running example:

- $|\mathfrak{M}| = \{0, 1, 2, 3, 4, 5\}$
- $a^{\mathfrak{M}} = 3$
- $s^{\mathfrak{M}} : \{0, 1, 2, 3, 4, 5\} \to \{0, 1, 2, 3, 4, 5\}$ maps 0 to 0, 1 to 2, 2 to 4, 3 to 0, 4 to 2, 5 to 4
- $P^{\mathfrak{M}} = \{(1), (2), (3)\}$, identified with $\{1, 2, 3\}$
- $R^{\mathfrak{M}} = \{(3, 0), (3, 1), (1, 5), (5, 1), (2, 2)\}$

Meaning of closed terms and formulas

Let $\mathfrak{M}$ be a structure over a vocabulary V.

- The meaning, or interpretation, in $\mathfrak{M}$ of a *closed* term over V is a member of $|\mathfrak{M}|$.
- The meaning, or interpretation, in $\mathfrak{M}$ of a *closed* formula over V is either "true" or "false".

To define formally the meaning of closed terms and formulas in structure $\mathfrak{M}$, it is necessary to be able to refer to each member of $|\mathfrak{M}|$, which is not possible in general.

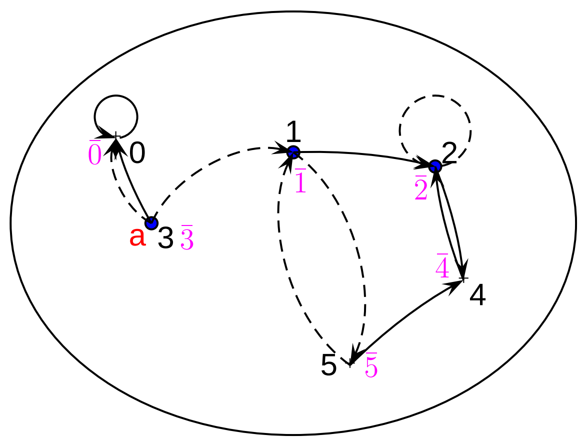
So we enrich the vocabulary with new constants, one per individual in $\mathfrak{M}$, defining:

$$V^{\mathfrak{M}} = V \cup \{\bar{c} \,|\, c \in |\mathfrak{M}|\}$$

where for each $c \in |\mathfrak{M}|$, $\bar{c}$ is a new constant.

$\mathfrak{M}$ now viewed as a structure over $V^{\mathfrak{M}}$

With our running example, $V^{\mathfrak{M}} = \{a, s, P, R, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$



Interpretation of closed terms

Note that every term over V is a term over $V^{\mathfrak{M}}$.

Given a closed term t over $V^{\mathfrak{M}}$, the interpretation of t in $\mathfrak{M}$ is denoted $t^{\mathfrak{M}}$. It is inductively defined as follows.

- For all $c \in |\mathfrak{M}|$, the interpretation $\bar{c}^{\mathfrak{M}}$ of $\bar{c}$ in $\mathfrak{M}$ is c.
- For all n-ary function symbols f in V and closed terms $t_1, \ldots, t_n$ over $V^{\mathfrak{M}}$, the interpretation $f(t_1, \ldots, t_n)^{\mathfrak{M}}$ of $f(t_1, \ldots, t_n)$ in $\mathfrak{M}$ is $f^{\mathfrak{M}}(t_1^{\mathfrak{M}}, \ldots, t_n^{\mathfrak{M}})$.

Interpretation of some terms in $\mathfrak{M}$

With our running example:

- $\bar{3}^{\mathfrak{M}} = 3$
- $a^{\mathfrak{M}} = 3$
- $s(s(s(a)))^{\mathfrak{M}} = s^{\mathfrak{M}}(s(s(a))^{\mathfrak{M}}) = s^{\mathfrak{M}}(s^{\mathfrak{M}}(s(a))^{\mathfrak{M}}) = s^{\mathfrak{M}}(s^{\mathfrak{M}}(s^{\mathfrak{M}}(a^{\mathfrak{M}}))) = s^{\mathfrak{M}}(s^{\mathfrak{M}}(s^{\mathfrak{M}}(3))) = s^{\mathfrak{M}}(s^{\mathfrak{M}}(0)) = s^{\mathfrak{M}}(0) = 0$
- $s(s(s(\bar{1})))^{\mathfrak{M}} = s^{\mathfrak{M}}(s(s(\bar{1}))^{\mathfrak{M}}) = s^{\mathfrak{M}}(s^{\mathfrak{M}}(s(\bar{1}))^{\mathfrak{M}}) = s^{\mathfrak{M}}(s^{\mathfrak{M}}(s^{\mathfrak{M}}(\bar{1}^{\mathfrak{M}}))) = s^{\mathfrak{M}}(s^{\mathfrak{M}}(s^{\mathfrak{M}}(1))) = s^{\mathfrak{M}}(s^{\mathfrak{M}}(2)) = s^{\mathfrak{M}}(4) = 2$

Interpretation of closed formulas (1)

Given a closed formula φ over $V^{\mathfrak{M}}$, the interpretation of φ in $\mathfrak{M}$ is either "true" or "false".
Note that every formula over V is a formula over $V^{\mathfrak{M}}$.

We write $\mathfrak{M} \models \varphi$ when φ is true in $\mathfrak{M}$ and $\mathfrak{M} \not\models \varphi$ otherwise.

The definition of "φ is true in $\mathfrak{M}$" is inductive:

- For all n-ary predicate symbols R in V and closed terms $t_1, \ldots, t_n$ over $V^{\mathfrak{M}}$, $\mathfrak{M} \models R(t_1, \ldots, t_n)$ iff $(t_1^{\mathfrak{M}}, \ldots, t_n^{\mathfrak{M}}) \in R^{\mathfrak{M}}$
- For languages with equality only: for all closed terms t_1, t_2, $\mathfrak{M} \models t_1 = t_2$ iff $t_1^{\mathfrak{M}} = t_2^{\mathfrak{M}}$

Interpretation of closed formulas (2)

- For all closed formulas ψ, ξ, $\mathfrak{M} \models \psi \vee \xi$ iff $\mathfrak{M} \models \psi$ or $\mathfrak{M} \models \xi$
- For all closed formulas ψ, ξ, $\mathfrak{M} \models \psi \wedge \xi$ iff $\mathfrak{M} \models \psi$ and $\mathfrak{M} \models \xi$
- For all closed formulas ψ, ξ, $\mathfrak{M} \models \psi \rightarrow \xi$ iff $\mathfrak{M} \not\models \psi$ or $\mathfrak{M} \models \xi$
- For all closed formulas ψ, ξ, $\mathfrak{M} \models \psi \leftrightarrow \xi$ iff both $\mathfrak{M} \models \psi$ and $\mathfrak{M} \models \xi$, or both $\mathfrak{M} \not\models \psi$ and $\mathfrak{M} \not\models \xi$
- For all formulas ψ and variables x such that $\exists x \psi$ is closed, $\mathfrak{M} \models \exists x \psi$ iff $\mathfrak{M} \models \psi[\bar{c}/x]$ for at least one $c \in |\mathfrak{M}|$
- For all formulas ψ and variables x such that $\forall x \psi$ is closed, $\mathfrak{M} \models \forall x \psi$ iff $\mathfrak{M} \models \psi[\bar{c}/x]$ for all $c \in |\mathfrak{M}|$

Interpretation of some formulas in $\mathfrak{M}$

With our running example:

- $\mathfrak{M} \models \neg R(s(s(s(\bar{1}))), a)$ since $s(s(s(\bar{1})))^{\mathfrak{M}} = 2$, $a^{\mathfrak{M}} = 3$, and $(2,3) \notin R^{\mathfrak{M}}$
- $\mathfrak{M} \models \exists x\, R(x, s(x))$ since $s(\bar{3})^{\mathfrak{M}} = 0$ and $(3,0) \in R^{\mathfrak{M}}$
- $\mathfrak{M} \models \exists x\, \neg R(x, s(x))$ since $s(\bar{2})^{\mathfrak{M}} = 4$ and $(2,4) \notin R^{\mathfrak{M}}$
- $\mathfrak{M} \models \forall x(P(x) \rightarrow \exists y R(x,y))$ since $P^{\mathfrak{M}} = \{1,2,3\}$, $(1,5) \in R^{\mathfrak{M}}$, $(2,2) \in R^{\mathfrak{M}}$, and $(3,1) \in R^{\mathfrak{M}}$
- $\mathfrak{M} \models \forall x \forall z(\exists y(R(x,y) \wedge R(y,z)) \rightarrow (P(x) \vee P(z)))$ as can be checked easily, using in particular the facts that $(3,1) \in R^{\mathfrak{M}}$, $(1,5) \in R^{\mathfrak{M}}$, and $3 \in P^{\mathfrak{M}}$.
- $\mathfrak{M} \models \forall x(\neg(P(s(x)) \vee P(s(s(x)))) \rightarrow (x = a \vee x = s(a)))$