Lecture notes 15.0

First-order logic: logical equivalence, satisfiability, validity, logical consequence

COMP 2411, session 1, 2004

Introduction

The notions of logical equivalence, satisfiability, validity, and logical consequence are defined in first-order logic in a similar way to propositional logic.

Indeed, they are all derived from the notion of truth (of a formula) in an interpretation.

The notion of interpretation is different in both cases: assignment of truth values to propositional letters in one case, and structures in the other.

These differences are basically irrelevant when it comes to the previous notions.

Models

We have defined the truth of a closed formula, also called a sentence, in a structure.

By convention, a formula that is not closed is true in a structure $\mathfrak{M}$ iff (any of) its universal closures are true in $\mathfrak{M}$.

For instance, with a vocabulary containing a binary predicate symbol P, the (nonclosed) formula $\exists x P(x,u) \to \forall z P(y,u)$ is true in a structure $\mathfrak{M}$ iff $\forall u \forall y (\exists x P(x,u) \to \forall z P(y,u))$ is true in $\mathfrak{M}$.

When a formula φ is true in a structure $\mathfrak{M}$, *i.e.*, when $\mathfrak{M} \models \varphi$, we say that $\mathfrak{M}$ is a model of φ.

Similarly, given a set of formulas T, we say that $\mathfrak{M}$ *is a model of* T, and we write $\mathfrak{M} \models T$, when $\mathfrak{M}$ is a model of all formulas in T.

Logical equivalence, validity

Two formulas are logically equivalent iff they have the same meaning in any structure. Formally, the formulas φ and ψ are logically equivalent just in case:

for all structures $\mathfrak{M}$, $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \psi$.

We say that a formula φ is valid, or a tautology, denoted $\models \varphi$, iff φ is true in all structures. Formally, the formula φ is valid, or a tautology, just in case:

for all structures $\mathfrak{M}$, $\mathfrak{M} \models \varphi$

Example of valid formulas include:

- $\forall x(x = x)$
- $\exists x(x = x)$
- $\forall x P(x) \to \exists x P(x)$.

Logical implication

Given two formulas φ and ψ, we say that ψ logically implies φ, or that φ is a logical consequence of ψ, denoted $\psi \models \varphi$, iff every model of ψ is a model of φ. So $\psi \models \varphi$ just in case:

for all structures $\mathfrak{M}$, if $\mathfrak{M} \models \psi$ then $\mathfrak{M} \models \varphi$.

Similarly, given a set of formulas T and a formula φ, we say that T logically implies φ, or that φ is a logical consequence of T, denoted $T \models \varphi$, iff every model of T is a model of φ. So $T \models \varphi$ just in case:

for all structures $\mathfrak{M}$, if $\mathfrak{M} \models T$ then $\mathfrak{M} \models \varphi$.

For instance:

- $\exists x \forall y (x = y) \models \forall x P(x) \vee \forall x \neg P(x)$
- $\{\forall x (P(x) \rightarrow Q(x)), \exists x P(x)\} \models \exists x Q(x)$

Satisfiability, validity, consistency

A formula φ is satisfiable iff it has a model; otherwise φ is said to be unsatisfiable.

A set of formulas T is consistent iff it has a model; otherwise T is said to be inconsistent.

For instance:

- $\exists x P(x) \wedge \exists x \neg P(x)$ is satisfiable.
- $\exists x (P(x) \wedge \neg P(x))$ is unsatisfiable.
- $\exists x (x \neq x)$ (abbreviation for $\exists x \neg (x = x)$) is unsatisfiable.
- $\{\varphi_1, \ldots, \varphi_n\}$ is consistent iff $\varphi_1 \wedge \ldots \wedge \varphi_n$ is satisfiable.
- $\emptyset$ is consistent.
- $\{\exists x P(x), \forall x \neg P(x)\}$ is inconsistent.

Basic properties (1)

Property: Every formula is a logical consequence of an inconsistent set of formulas.

Property: For all closed formulas φ, the following are equivalent.

- φ is valid.
- $\neg\varphi$ is unsatisfiable.
- φ is a logical consequence of the empty set.

Property: For all closed formulas ψ and φ, the following are equivalent.

- φ is a logical consequence of ψ.
- $\psi \rightarrow \varphi$ is valid.
- $\psi \wedge \neg\varphi$ is unsatisfiable.

Basic properties (2)

Property: For all sets of formulas T and for all closed formulas φ, the following are equivalent.

- φ is a logical consequence of T.
- $T \cup \{\neg\varphi\}$ is inconsistent.

Property: For all closed formulas ψ and φ, the following are equivalent.

- φ is a logical consequence of ψ and ψ is a logical consequence of φ.
- φ and ψ are logical equivalent.
- $\psi \leftrightarrow \varphi$ is valid.

Logical equivalences (1)

The logical equivalences of propositional logic are obviously equivalences of first-order logic.

To be understood, and by no means learnt by heart, are the following logical equivalences involving quantifiers.

- $\neg\exists x\varphi$ is logically equivalent to $\forall x\neg\varphi$
- $\neg\forall x\varphi$ is logically equivalent to $\exists x\neg\varphi$
- $\exists x\exists y\varphi$ is logically equivalent to $\exists y\exists x\varphi$
- $\forall x\forall y\varphi$ is logically equivalent to $\forall y\forall x\varphi$
- $\exists x\forall y\varphi$ logically implies $\forall y\exists x\varphi$

Logical equivalences (2)

- $\exists x\varphi$ is logically equivalent to φ if there is no free occurrence of x in φ
- $\forall x\varphi$ is logically equivalent to φ if there is no free occurrence of x in φ
- $\forall x(\varphi \wedge \psi)$ is logically equivalent to $\forall x\varphi \wedge \forall x\psi$
- $\exists x(\varphi \vee \psi)$ is logically equivalent to $\exists x\varphi \vee \exists x\psi$
- $\forall x(\varphi \vee \psi)$ is logically equivalent to $\forall x\varphi \vee \psi$ if there is no free occurrence of x in ψ
- $\exists x(\varphi \wedge \psi)$ is logically equivalent to $\exists x\varphi \wedge \psi$ if there is no free occurrence of x in ψ

We sometimes write $\varphi \equiv \psi$ to denote that φ and ψ are logically equivalent.

Renaming of variables (1)

Given a formula φ, a variable x having free occurrences in φ and a term t, it is always possible to rename the bound occurrences of variables in φ, and get a formula $\widehat{\varphi}$ such that:

- no variable has both free and bound occurrences in $\widehat{\varphi}$;
- no variable has both an occurrence in t and a bound occurrence in $\widehat{\varphi}$
- occurrences of variables in $\widehat{\varphi}$ that immediately follow distinct occurrences of quantifiers are not occurrences of the same variable;
- t is substitutible for x and $\widehat{\varphi}$;
- $\varphi \leftrightarrow \widehat{\varphi}$ is valid.

The first three conditions above basically guarantee ‘good readability’.

Renaming of variables (2)

For instance, take

$$\varphi = \exists x P(x,y) \wedge \neg\exists z\forall y(Q(x,y,z) \vee \exists x R(x,z))$$

and

$$t = g(v,x,y).$$

If we want to substitute y by t in a formula $\widehat{\varphi}$ that satisfies the conditions above with y playing the role of x, we can define $\widehat{\varphi}$ as:

$$\exists u P(u,y) \wedge \neg\exists z\forall t(Q(x,t,z) \vee \exists w R(w,z))$$

Properties of equality

In the following, t, t_1, t_2, t_3 denote terms.

- $t = t$ is a valid.
- $t_2 = t_1$ is a logical consequence of $t_1 = t_2$
- $t_3 = t_1$ is a logical consequence of $(t_1 = t_2) \wedge (t_2 = t_3)$
- Given a formula φ and a variable x, if t_1 and t_2 are both substitutible for x in φ then $t_1 = t_2$ logically implies $\varphi[t_1/x] \leftrightarrow \varphi[t_2/x]$