Lecture notes 17.1

Predicate calculus: Hilbert systems

COMP 2411, session 1, 2004

Introduction

We extend the Hilbert system for propositional logic to first-order logic.

The primitive operators are negation, implication, and the universal quantifier.

We view:

- $\varphi \vee \psi$ as an abbreviation for $\neg\varphi \to \psi$;
- $\varphi \wedge \psi$ as an abbreviation for $\neg(\varphi \to \neg\psi)$;
- $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \to \psi) \wedge (\psi \to \varphi)$, hence as an abbreviation for $\neg((\varphi \to \psi) \to \neg(\psi \to \varphi))$;
- $\exists x \varphi$ is an abbreviation for $\neg \forall x \neg \varphi$.

For simplicity we consider vocabularies without equality and with constants as only function symbols.

Axioms

We add two axiom schemes to the three axiom schemes of propositional logic and get:

Axiom 1: $\varphi \to \psi \to \varphi$

Axiom 2: $(\varphi \to \psi \to \xi) \to (\varphi \to \psi) \to \varphi \to \xi$

Axiom 3: $(\neg\psi \to \neg\varphi) \to (\neg\psi \to \varphi) \to \psi$

Axiom 4: $\forall x\, \varphi \to \varphi[a/x]$ where a is a constant.

Axiom 4: $\forall x (\varphi \to \psi) \to \varphi \to \forall x\, \psi$ provided that x has no free occurrence in φ.

It is immediately verified that (every instance of) theses axioms is valid.

Rules of inference (1)

Beware that the textbook contains serious mistakes in this part.

We add an inference rule to modus ponens and get:

Modus ponens:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

Generalization:

$$\frac{\varphi[a/x]}{\forall x\, \varphi}$$

where a is a constant.

The generalization rule is not valid: a model of $p(a)$ is not necessarily a model of $\forall x\, p(x)$.

Rules of inference (2)

The deduction rule can be derived from the previous axioms and rules of inference, in the following form.

Deduction rule:

$$\frac{X \cup \{\varphi\} \vdash \psi}{X \vdash \varphi \rightarrow \psi}$$

provided that φ does not contain any constant previously used in the generalization rule.

The notation $X \vdash \varphi$ means that φ has been derived from the set of assumptions X.

These axioms and rules of inference do not determine a sound and complete proof procedure for logical consequence ($X \models \varphi$), but they do determine a sound and complete proof procedure for validity ($\models \varphi$).

Rules of inference (3)

Intuitively, if we try and prove that φ is valid, then the constant a in the generalization rule can only represent an arbitrary individual, and the generalization to $\forall x\, \varphi$ is legitimate (in contrast, if we had to show that $X \models \varphi$ and if a occurred in X, then a would denote a specific individual and the generalization would be faulty).

The proviso on the deduction rule achieves the same effect on extra (provisonal) hypotheses. Without it, we could write:

1.	$p(a)$	(Extra) hypothesis
2.	$\forall x\, p(x)$	Generalization
3.	$p(a) \rightarrow \forall x\, p(x)$	WRONG Deduction 2, lift 1

and (incorrectly) conclude that $p(a) \rightarrow \forall x\, p(x)$ is valid.

Remark

It is possible to change the previous inference rules and get a sound and complete proof system for logical consequence, and not only for validity.

Basically, constants should be replaced by variables in the generalization rule.

These are technical details of little consequence, since $\{\psi_1, \ldots, \psi_n\} \models \varphi$ iff $\psi_1 \wedge \ldots \wedge \psi_n \rightarrow \varphi$ is valid, hence there is no significant loss in generality in using constants rather than variables.

Example 1

The following is an example of proof (together with justifications). It shows that $\forall x(p(x) \rightarrow q(x)) \rightarrow \forall x\, p(x) \rightarrow \forall x\, q(x)$ is valid.

1.	$\forall x\, p(x)$	(Extra) hypothesis
2.	$\forall x\, p(x) \rightarrow p(a)$	Axiom 4
3.	$p(a)$	MP 2,1
4.	$\forall x(p(x) \rightarrow q(x))$	(Extra) hypothesis
5.	$\forall x(p(x) \rightarrow q(x)) \rightarrow p(a) \rightarrow q(a)$	Axiom 4
6.	$p(a) \rightarrow q(a)$	MP 5,4
7.	$q(a)$	MP 6,3
8.	$\forall x\, q(x)$	Generalization 7
9.	$\forall x\, p(x) \rightarrow \forall x\, q(x)$	Deduction 8, lift 1
10.	$\forall x(p(x) \rightarrow q(x)) \rightarrow \forall x\, p(x) \rightarrow \forall x\, q(x)$	Deduction 9, lift 4

Example 2

The easiest way to show that $\forall x\, p(x) \to \forall y\, p(y)$ is valid is with the deduction rule.

1.	$\forall x\, p(x)$	(Extra) hypothesis
2.	$\forall x\, p(x) \to p(a)$	Axiom 4
3.	$p(a)$	MP 2,1
4.	$\forall y\, p(y)$	Generalization 3
5.	$\forall x\, p(x) \to \forall y\, p(y)$	Deduction 4, lift 1

Example 3

Here is an alternative and even shorter proof that $\forall x\, p(x) \to \forall y\, p(y)$ is valid that does not use the deduction rule.

1.	$\forall x\, p(x) \to p(a)$	Axiom 4
2.	$\forall y (\forall x\, p(x) \to p(y))$	Generalization 1
3.	$\forall y (\forall x\, p(x) \to p(y)) \to \forall x\, p(x) \to \forall y\, p(y)$	Axiom 5
4.	$\forall x\, p(x) \to \forall y\, p(y)$	MP 3,2

Proof: general definition

Given a formula φ, a proof of φ (in the proof-system we are investigating) is a finite sequence $(\varphi_0, \ldots, \varphi_n)$ of formulas such that:

- $\varphi_n = \varphi$ (the proof ends with the formula to be proved).
- For all $k \leq n$, either:
  - φ_k is an instance of one of the five axiom schemes, or
  - φ_k follows from φ_i and φ_j with $i, j < k$ (two formulas that occur before φ_k in the proof) using modus ponens, or
  - φ_k follows from φ_i with $i < k$ (one formula that occurs before φ_k in the proof) using the generalization rule.

The deduction rule is a derived rule, that is not necessary (but that is still extremely convenient).

Abbreviated proofs

We are mainly interested in quantifiers, we know that the first three axioms and MP provide a sound a complete proof procedure for propositional logic, and that writing down proofs in this system is highly frustrating.

A compromise is to write down abbreviated proofs where all the sequences of steps that can be justified in propositional logic are skipped.

We give a couple of examples of such proofs.

Example 4

The following is an example of an abbreviated proof that $\forall x\, p(x) \to \exists x\, p(x)$ is valid.

1.	$\forall x\, p(x)$	(Extra) hypothesis
2.	$\forall x\, p(x) \to p(a)$	Axiom 4
3.	$p(a)$	MP 2,1
4.	$\forall x \neg p(x) \to \neg p(a)$	Axiom 4
5.	$p(a) \to \neg \forall x \neg p(x)$	Prop. logic 4
6.	$\neg \forall x \neg p(x)$	MP 5,3
7.	$\forall x\, p(x) \to \neg \forall x \neg p(x)$	Deduction 6, lift 1

Remember that $\exists x\, p(x)$ is *defined* here as an abbreviation for $\neg \forall x \neg p(x)$, hence we are done.

Example 5

The following is an example of an abbreviated proof that $\forall x (p(a) \to q(x)) \leftrightarrow (p(a) \to \forall x\, q(x))$ is valid.

1.	$p(a) \to \forall x\, q(x)$	(Extra) hypothesis
2.	$\forall x\, q(x) \to q(b)$	Axiom 4
3.	$p(a) \to q(b)$	Prop. logic 1,2
4.	$\forall x (p(a) \to q(x))$	Generalization 3
5.	$(p(a) \to \forall x\, q(x)) \to \forall x (p(a) \to q(x))$	Deduction 4, lift 1
6.	$\forall x (p(a) \to q(x)) \to p(a) \to \forall x\, q(x)$	Axiom 5
7.	$\forall x (p(a) \to q(x)) \leftrightarrow (p(a) \to \forall x\, q(x))$	Prop. logic 5,6

Soundness and completeness

The proof system we have defined is sound and complete for validity:

Proposition: For all formulas φ, $\models \varphi$ iff there exists a proof of φ.

Intuitively, this means that:

- axioms and rules of inference are correct (soundess);
- there are enough axioms and rules of inference to prove φ, for all valid formulas φ.