Lecture notes 18.1

First-order logic: natural deduction

COMP 2411, session 1, 2004

Introduction

Natural deduction for first-order logic uses the rules of inference of propositional logic, plus inference rules for the quantifiers and in case of languages with equality, inference rules for equality.

As for the boolean operators, there is for each quantifier an introduction rule (I) and an elimination rule (E).

For equality on the other hand, the rules are more like substitution rules.

Rules for $\forall$ (1)

$\forall$ elimination:

If $\forall x \varphi$ has been derived and if term t is free for x in φ, then it is legitimate to derive $\varphi[t/x]$:

$$\frac{\forall x \varphi}{\varphi[t/x]} \ \forall E$$

Rules for $\forall$ (2)

$\forall$ introduction:

If φ has been derived and if, for any hypothesis χ used in the derivation of φ, variable x does not occur free in χ, then it is legitimate to derive $\forall x \varphi$. So provided that the conditions just stated are fulfilled, we can apply the following rule of inference:

$$\frac{\varphi}{\forall x \varphi} \ \forall I$$

The intuitive interpretation of the previous rule is the following. Suppose that x satisfies the conditions of application of the rule. Then x denotes an arbitrary individual, and the generalization to $\forall x \varphi$ is legitimate.

Example 1

$\forall x(\varphi \wedge \psi) \rightarrow (\forall x\varphi \wedge \forall x\psi)$ is a valid:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\varphi \wedge \psi)]^1}{\varphi \wedge \psi}\;\forall E}{\varphi}\;\wedge E}{\forall x\varphi}\;\forall I \qquad \dfrac{\dfrac{\dfrac{[\forall x(\varphi \wedge \psi)]^1}{\varphi \wedge \psi}\;\forall E}{\psi}\;\wedge E}{\forall x\psi}\;\forall I}{\forall x\varphi \wedge \forall x\psi}\;\wedge I}{\forall x(\varphi \wedge \psi) \rightarrow (\forall x\varphi \wedge \forall x\psi)}\;{\rightarrow} I_1$$

Example 2

If x does not occur free in φ, then $\varphi \rightarrow \forall x\psi$ is a logical consequence of $\forall x(\varphi \rightarrow \psi)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x(\varphi \rightarrow \psi)}{\varphi \rightarrow \psi}\;\forall E \qquad [\varphi]^1}{\psi}\;{\rightarrow} E}{\forall x\psi}\;\forall I}{\varphi \rightarrow \forall x\psi}\;{\rightarrow} I_1$$

Example 3

The following is not a correct proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{[x = 0]^1}{\forall x(x = 0)}\;\forall I}{x = 0 \rightarrow \forall x(x = 0)}\;{\rightarrow} I_1}{\forall x(x = 0 \rightarrow \forall x(x = 0))}\;\forall I}{0 = 0 \rightarrow \forall x(x = 0)}\;\forall E$$

The first application of $\forall I$ is illegal, because x is free in $x = 0$.

Indeed, $0 = 0 \rightarrow \forall x(x = 0)$ is not valid: it is false in any structure that contains at least two individuals.

Rules for $\exists$ (1)

$\exists$ elimination: If

- $\exists x\varphi$ has been derived;
- assuming φ, ψ can be derived, with the condition that for any hypothesis χ used in the derivation of ψ that is distinct from φ, variable x does not occur free in χ

then it is legitimate to derive ψ, removing the assumption that φ holds (indicated by putting φ between square brackets). So provided that the conditions just stated are fulfilled, we can apply the following inference rule:

$$\dfrac{\exists x\varphi \qquad \begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix}}{\psi}\;\exists E$$

Rules for $\exists$ (2)

The intuitive interpretation of the previous rule is:

- We know that property φ holds for some individual.
- Let us pick up such an individual (denoted by x).
- If property ψ can be inferred, and if neither ψ nor any assumption in the proof that ψ holds says anything about the object denoted by x, then we can derive ψ.

$\exists$ introduction:

If term t is free for x in φ and if $\varphi[t/x]$ has been derived, then it is legitimate to derive $\exists x\varphi$:

$$\frac{\varphi[t/x]}{\exists x\varphi}\ \exists I$$

Example 4

$\exists x\varphi \vee \exists x\psi$ is a logical consequence of $\exists x(\varphi \vee \psi)$:

$$\dfrac{\exists x(\varphi \vee \psi) \qquad \dfrac{[\varphi \vee \psi]^2 \qquad \dfrac{\dfrac{[\varphi]^1}{\exists x\varphi}\ \exists I}{\exists x\varphi \vee \exists x\psi}\ \vee I \qquad \dfrac{\dfrac{[\psi]^1}{\exists x\psi}\ \exists I}{\exists x\varphi \vee \exists x\psi}\ \vee I}{\exists x\varphi \vee \exists x\psi}\ \vee E_1}{\exists x\varphi \vee \exists x\psi}\ \exists E_2$$

Rules for equality (1)

We have to express that equality is an equivalence relation, *i.e.*, is reflexive, symmetric, and transitive.

For reflexivity, we have a rule with no premise, *i.e.*, an axiom scheme (an axiom for each variable):

$$\frac{}{x = x}\ Eq1$$

The following inference rule is for symmetry:

$$\frac{x = y}{y = x}\ Eq2$$

The following inference rule is for transitivity:

$$\frac{x = y \quad y = z}{x = z}\ Eq3$$

Rules for equality (2)

Consider a term t built from a function symbol and variables.

If we replace in t a variable x by a variable y and if x and y denote the same individual, then the resulting term denotes the same individual as t.

Generalized to many variables, this is captured by the following inference rules, one for each n-ary function symbol f in the vocabulary:

$$\frac{x_1 = y_1 \quad \ldots \quad x_n = y_n}{f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)}\ Eq4$$

Rules for equality (3)

Consider an atomic formula φ built from a predicate symbol and variables.

If we replace in φ a variable x by a variable y and if x and y denote the same individual, then the resulting formula is true whenever φ is true.

Generalized to many variables, this is captured by the following inference rules, one for each n-ary predicate symbol p in the vocabulary:

$$\frac{x_1 = y_1 \quad \ldots \quad x_n = y_n \quad p(x_1, \ldots, x_n)}{p(y_1, \ldots, y_n)}\ Eq5$$

Alternative rules for $=$ (1)

The rules we have presented are powerful enough to be the basis of a sound a complete proof system.

The rules for equality are very ‘low level’ and could advantageously replaced by more powerful inference rules (that can be derived from the simpler ones we have described).

For instance, we could express that when we substitute in a term some occurrences of a variable by equal variables, then we get equal terms.

Generalized to many variables, this would be captured by the following inference rule:

$$\frac{x_1 = y_1 \quad \ldots \quad x_n = y_n}{t[x_1/z_1, \ldots, x_n/z_n] = t[y_1/z_1, \ldots, y_n/z_n]}\ Eq4'$$

Example 5

Let f and g be two unary function symbols.

Rule *Eq4'* would enable to derive in one step that $f(g(x)) = f(g(y))$ is a logical consequence of $x = y$.

With the primitive rules we have introduced, more work needs to be done:

$$\begin{prooftree}
\AxiomC{$[x = y]^1$}
\RightLabel{$Eq4$}
\UnaryInfC{$f(x) = f(y)$}
\RightLabel{$\to I_1$}
\UnaryInfC{$x = y \to f(x) = f(y)$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall y(x = y \to f(x) = f(y))$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x \forall y(x = y \to f(x) = f(y))$}
\RightLabel{$\forall E$}
\UnaryInfC{$\forall y(g(x) = y \to f(g(x)) = f(y))$}
\RightLabel{$\forall E$}
\UnaryInfC{$g(x) = g(y) \to f(g(x)) = f(g(y))$}
\AxiomC{$x = y$}
\RightLabel{$Eq4$}
\UnaryInfC{$g(x) = g(y)$}
\RightLabel{$\to E$}
\BinaryInfC{$f(g(x)) = f(g(y))$}
\end{prooftree}$$

Alternative rules for $=$ (2)

Similarly, we could express that when we substitute in a formula φ some occurrences of a variable by equal variables, then we get a formula that is a logical consequence of φ.

Generalized to many variables, this would be captured by the following inference rule:

$$\frac{x_1 = y_1 \quad \ldots \quad x_n = y_n \quad \varphi[x_1/z_1, \ldots, x_n/z_n]}{\varphi[y_1/z_1, \ldots, y_n/z_n]}\ Eq5'$$