Lecture notes 4.0

Propositional calculus: satisfiability, validity, logical consequence

COMP 2411, session 1, 2004

Validity, satisfiability (1)

When we build a truth table for a propositional formula φ, we obtain one and only one of the following 3 cases:

1. the last column is filled with true only (*e.g.*, $\varphi = p \vee \neg p$), or
2. the last column is filled with false only (*e.g.*, $\varphi = p \wedge \neg p$), or
3. the last column is filled with both true and false (*e.g.*, $\varphi = p$).

The notions of satisfiability and validity enable to describe these cases.

- Definition: A formula is satisfiable if it is true in some interpretation.
- Definition: A formula is valid if it is true in all interpretations.

Validity, satisfiability (2)

A valid formula is also called a tautology.

From these definitions we obtain:

- A valid formula is satisfiable.
- A formula φ is valid iff $\neg\varphi$ is not satisfiable.
- A formula φ is satisfiable iff $\neg\varphi$ is not valid.

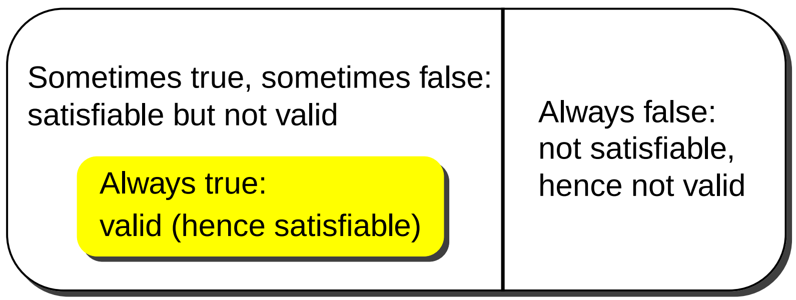
1. In the first case above, φ is valid (*e.g.*, $\varphi = p \vee \neg p$ is valid)—hence also satisfiable.
2. In the second case above, φ is not satisfiable (*e.g.*, $\varphi = p \wedge \neg p$ is not satisfiable)—hence also not valid.
3. In the third case above, both φ and $\neg\varphi$ are satisfiable (*e.g.*, $\varphi = p$ and $\neg\varphi = \neg p$ are both satisfiable)—hence also neither φ nor $\neg\varphi$ is valid.

Validity, satisfiability (3)

Validity captures the notion of being logically true—true by rational necessity.

Satisfiability captures the notion of being sometimes true—possibly true.

Both notions can be depicted as follows.



Models

Definition: An interpretation in which a formula φ is true is called a model of φ.

Hence a formula has a model iff it is satisfiable.

For example, any interpretation thats assigns true to p is a model of $p \vee \neg q \wedge r$, whereas any interpretation that assigns false to p and true to q is **not** a model of $p \vee \neg q \wedge r$.

Definition: A model of a set X of formulas is a model of all members of X.

For example, any interpretation that assigns true to p, true to q and false to r is a model of $\{p, p \rightarrow q, \neg p \vee \neg r\}$, whereas any interpretation that assigns true to p and true to r is **not** a model of $\{p, p \rightarrow q, \neg p \vee \neg r\}$

Logical consequence (1)

Rational reasoning is captured by the following notion.

Definition: A formula φ is a logical consequence of a set X of formulas iff every model of X is a model of φ.

If φ is a logical consequence of X then we write $X \models \varphi$; otherwise we write $X \not\models \varphi$.

For example: $\{p, \neg q\} \models \{(p \vee r) \wedge (\neg q \vee \neg r)\}$

Intuitively, if φ is a logical consequence of X, then φ is logically, necessarily true, under the assumption that all members of X are true.

We write $\psi \models \varphi$ rather than $\{\psi\} \models \varphi$; for instance, we write $p \wedge q \models p$ rather than $\{p \wedge q\} \models p$.

Logical consequence (2)

Logical consequence and validity are closely related:

Property: For all $n \in \mathbb{N}$ and formulas $\psi_1, \ldots, \psi_n, \varphi$, φ is a logical consequence of $\{\psi_1, \ldots, \psi_n\}$ iff $\psi_1 \wedge \ldots \wedge \psi_n \rightarrow \varphi$ is valid.

Note the particular case where $n = 0$: it yields that φ is a logical consequence of the empty set—also written $\models \varphi$ rather than $\emptyset \models \varphi$—iff φ is valid.

The notion of logical consequence might look more general than the notion of validity, due to infinite sets of premises: indeed, we cannot write $\{\psi_1, \psi_2, \psi_3, \ldots\} \models \varphi$ iff $(\psi_1 \wedge \psi_2 \wedge \ldots) \rightarrow \varphi$ is valid because infinite conjunctions are not allowed.

Compactness

The compactness theorem shows that actually, logical consequence can always be reduced to validity:

Proposition: For all (possible infinite) sets X of formulas and for all formulas φ, $X \models \varphi$ iff $D \models \varphi$ for some finite subset D of X.

For instance, $\{p_1 \rightarrow p_2,\ p_2 \rightarrow p_3,\ \ldots\} \models p_2 \rightarrow p_6$, but also

$$\{p_2 \rightarrow p_3,\ p_3 \rightarrow p_4,\ p_4 \rightarrow p_5,\ p_5 \rightarrow p_6\} \models p_2 \rightarrow p_6$$

Hence:

Corollary: For all (possible infinite) sets X of formulas and for all formulas φ, $X \models \varphi$ iff $\psi_1 \wedge \ldots \wedge \psi_n \rightarrow \varphi$ is valid for some finite subset $\{\psi_1, \ldots, \psi_n\}$ of X.

Theories

Definition: A theory is a set of formulas that is closed under logical consequence.

Hence every theory is infinite.

Definition: A theory is consistent iff it has at least one model; otherwise the theory is inconsistent.

Property: Given a theory T, the following conditions are equivalent:

- T is inconsistent;
- there exists a formula φ such that both φ and $\neg\varphi$ belong to T;
- for all formulas φ, φ belongs to T.

In other words, contradictions are not local, but spread over the whole theory.