Lecture notes 7.2

Propositional calculus:
Hilbert systems

COMP 2411, session 1, 2004

Introduction

Let a formula φ be given.

To prove that φ is valid using semantic tableaux, we start from $\neg\varphi$, decompose it into smaller formulas, and check that no assignment of truth value to the constituting atoms can make $\neg\varphi$ true. This proof procedure is top-down.

Hilbert-style proof systems offer a bottom-up approach: starting from a list of axioms and applying inference rules in all possible ways, we try to generate φ;
φ is valid iff this attempt succeeds.

Proofs in the second type of system are not so easy to understand intuitively; but they give good insights into the distinction between proof and semantics.

Choice of language

We assume that $\neg$ and $\rightarrow$ are the only primitive operators of our language (remember that they can generate all boolean operators).

We view:

- $\varphi \vee \psi$ as an abbreviation for $\neg\varphi\rightarrow\psi$;
- $\varphi \wedge \psi$ as an abbreviation for $\neg(\varphi\rightarrow\neg\psi)$;
- $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi\rightarrow\psi) \wedge (\psi\rightarrow\varphi)$, hence as an abbreviation for:

$$\neg((\varphi\rightarrow\psi)\rightarrow\neg(\psi\rightarrow\varphi)).$$

Choosing a minimal syntax enables to keep axioms and rules of inference to a minimum.

Choice of axioms

There are infinitely many possible choices of axioms.

The selected axioms should just be valid formulas.

We choose the following axiom schemes, meaning that we can substitute for φ, ψ and ξ below any propositional formula in our language (where all formulas are built from atomic formulas using $\neg$ and $\rightarrow$ only).

Axiom 1: $\varphi\rightarrow\psi\rightarrow\varphi$

Axiom 2: $(\varphi\rightarrow\psi\rightarrow\xi)\rightarrow(\varphi\rightarrow\psi)\rightarrow\varphi\rightarrow\xi$

Axiom 3: $(\neg\psi\rightarrow\neg\varphi)\rightarrow(\neg\psi\rightarrow\varphi)\rightarrow\psi$

Choice of rules of inference

There are infinitely many possible choices of rules of inference.

The selected rules should just be valid, meaning that the conclusion of a rule should be a logical consequence of the premises of the rule.

We choose the following rule, where we can substitute for φ and ψ any propositional formula in our language (where all formulas are built from atomic formulas using $\neg$ and $\rightarrow$ only)

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$$

This rule is know as modus ponens (MP).

Example 1

The following is an example of proof (together with justifications).

The proof shows that $p \rightarrow p$ is valid.

1.	$(p \rightarrow (p \rightarrow p) \rightarrow p) \rightarrow (p \rightarrow p \rightarrow p) \rightarrow p \rightarrow p$	Axiom 2
2.	$p \rightarrow (p \rightarrow p) \rightarrow p$	Axiom 1
3.	$(p \rightarrow p \rightarrow p) \rightarrow p \rightarrow p$	MP 1,2
4.	$p \rightarrow p \rightarrow p$	Axiom 1
5.	$p \rightarrow p$	MP 3,4

Proof: general definition

Given a set X of formulas and a formula φ, a proof of φ from X (in the proof-system we are investigating) is a finite sequence $(\varphi_0, \ldots, \varphi_n)$ of formulas such that:

- $\varphi_n = \varphi$ (the proof ends with the formula to be proved).
- For all $k \leq n$, either:
  - φ_k belongs to X (is an assumption), or
  - φ_k is an instance of one of the three axiom schemes, or
  - φ_k follows from φ_i and φ_j with $i, j < k$ (two formulas that occur before φ_k in the proof) using modus ponens.

Example 2

The following is an example of a proof of q from $\{p, \neg p\}$ (together with justifications).

The proof shows that q is a logical consequence of $\{p, \neg p\}$.

1.	p	Hypothesis
2.	$\neg p$	Hypothesis
3.	$p \rightarrow \neg q \rightarrow p$	Axiom 1
4.	$\neg p \rightarrow \neg q \rightarrow \neg p$	Axiom 1
5.	$\neg q \rightarrow p$	MP 1,3
6.	$\neg q \rightarrow \neg p$	MP 2,4
7.	$(\neg q \rightarrow \neg p) \rightarrow (\neg q \rightarrow p) \rightarrow q$	Axiom 3
8.	$(\neg q \rightarrow p) \rightarrow q$	MP 6,7
9.	q	MP 5,8

Derived rules

Writing proofs in this system is rather frustrating. . .

Derived rules help a bit: they play the role of lemmas that can be used to shorten arguments.

A very useful derived rule is the deduction rule:

> Let a set X of formulas and two formulas ψ and φ be given. If there exists a proof of φ from $X \cup \{\psi\}$ then there exists a proof of $\psi \to \varphi$ from X.

The deduction rule involves extra or provisional hypotheses that are eventually lifted (removed) when the rule is applied.

See textbook for other examples of derived rules in a slightly different system (in axiom 3).

Example 3

The following is an example of a proof of $(\neg q \to \neg p) \to p \to q$ (from $\emptyset$, together with justifications), that uses the deduction rule.

1.	$\neg q \to \neg p$	Extra hypothesis
2.	$(\neg q \to \neg p) \to (\neg q \to p) \to q$	Axiom 3
3.	$p \to \neg q \to p$	Axiom 1
4.	$(\neg q \to p) \to q$	MP 1,2
5.	p	Extra hypothesis
6.	$\neg q \to p$	MP 5,3
7.	q	MP 6,4
8.	$p \to q$	Deduction rule 5,7, lift 5.
9.	$(\neg q \to \neg p) \to p \to q$	Deduction rule 1,8, lift 1.

Justification of deduction rule (1)

Consider a proof $\varphi_0, \ldots, \varphi_n$ of φ from $X \cup \{\psi\}$ (hence $\varphi_n = \varphi$). We have to build a proof of $\psi \to \varphi$ from X.

For that, we inductively build a proof of $\psi \to \varphi_k$ for all $k \leq n$.

Let $k \leq n$ be given, and suppose that we have a proof of $\psi \to \varphi_j$ for all $j < k$.

Case 1: Assume that φ_k is an axiom or a hypothesis in X. Then we obtain the following proof of $\psi \to \varphi_k$ (from $\emptyset$, hence from X).

1.	$\varphi_k \to \psi \to \varphi_k$	Axiom 1
2.	φ_k	Axiom or hypothesis
3.	$\psi \to \varphi_k$	MP 2,1

Justification of deduction rule (2)

Case 2: φ is ψ.

Then we have given a proof of (an instance of) $\varphi \to \varphi$ from $\emptyset$, hence also from X, on page 6.

Case 3: φ_k is obtained from two earlier formulas φ_i and φ_j, $i, j < k$, by modus ponens, and $\varphi_j = \varphi_i \to \varphi_k$.

By inductive hypothesis, we have a proof P_1 of $\psi \to \varphi_i$ and a proof P_2 of $\psi \to \varphi_i \to \varphi_k$ from X.

Justification of deduction rule (3)

Then we obtain the following proof of $\psi \rightarrow \varphi_k$ from X.

	$\vdots$	P_1 minus last line
m	$\psi \rightarrow \varphi_i$	
	$\vdots$	P_2 minus last line
n	$\psi \rightarrow \varphi_i \rightarrow \varphi_k$	
$n+1$	$(\psi \rightarrow \varphi_i \rightarrow \varphi_k) \rightarrow (\psi \rightarrow \varphi_i) \rightarrow \psi \rightarrow \varphi_k$	Axiom 2
$n+2$	$(\psi \rightarrow \varphi_i) \rightarrow \psi \rightarrow \varphi_k$	MP n,$n+1$
$n+3$	$\psi \rightarrow \varphi_k$	MP m,$n+2$

Soundness and completeness

The proof system we have defined is also sound and complete:

Proposition: For all sets of formulas X and for all formulas φ, $X \models \varphi$ iff there exists a proof of φ from X (see page 7).

Intuitively, this means that:

- axioms and rules of inference are correct (soundess);
- there are enough axioms and rules of inference to prove φ from X, for all X and φ with $X \models \varphi$.

To prove the proposition above, we can use the fact that the tableau construction offers a sound and complete proof procedure: it then suffices to 'mimic' every step in the tableau construction by a proof in the present proof system.