

Introduction

Lecture notes 8.2

Propositional logic: natural deduction

COMP 2411, session 1, 2004

Natural deduction is a sound and complete proof procedure that, like Hilbert-style systems, offers a bottom-up approach to proving: try and derive the formula to be proved from the set of assumptions and from the set of axioms, applying some rules of inference.

Whereas Hilbert-style systems have very few rules of inference (e.g., modus ponens only) and an obscure set of axioms, Natural deduction has no axiom but many rules of inference.

More precisely, for every boolean operator, Natural deduction has an **introduction rule** and an **elimination rule** that capture the meaning of the operator.

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Rules for conjunction

$\wedge$ introduction:

If both φ and ψ have been derived, then it is legitimate to derive $\varphi \wedge \psi$:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge I$$

$\wedge$ elimination:

If $\varphi \wedge \psi$ has been derived, then it is legitimate to derive both φ and ψ :

$$\frac{\varphi \wedge \psi}{\varphi} \wedge E$$

$$\frac{\varphi \wedge \psi}{\psi} \wedge E$$

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Rules for implication

$\rightarrow$ introduction:

If, assuming φ , ψ can be derived, then it is legitimate to derive $\varphi \rightarrow \psi$, removing the assumption that φ holds (indicated by putting φ between square brackets):

$$\frac{\begin{array}{c} [\varphi] \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \rightarrow I$$

$\rightarrow$ elimination:

If both φ and $\varphi \rightarrow \psi$ have been derived, then it is legitimate to derive ψ . This is nothing but modus ponens:

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \rightarrow E$$

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The $\perp$ symbol

Some rules involve a extra symbol: $\perp$.

This symbol is like an extra formula meaning ‘contradiction’; it could be replaced by an arbitrary unsatisfiable formula (*i.e.*, $p \wedge \neg p$).

The next two rules are rules for $\perp$ itself. The introduction rule for $\perp$ is denoted **RAA** rather than $\perp_E$.

RAA stands for *reductio ad absurdum*’ which means **reasoning by contradiction**.

Rules for $\perp$

$\perp$ introduction:

If, assuming $\neg\varphi$, a contradiction can be derived, then it is legitimate to derive φ , removing the assumption that $\neg\varphi$ holds (indicated by putting $\neg\varphi$ between square brackets):

$$\frac{[\neg\varphi] \dots \perp}{\varphi} \text{ RAA}$$

$\perp$ elimination:

Anything can be derived from a contradiction:

$$\frac{\perp}{\varphi} \perp$$

Rules for $\vee$ (1)

$\vee$ introduction:

If φ or ψ has been derived, then it is legitimate to derive $\varphi \vee \psi$:

$$\frac{\varphi}{\varphi \vee \psi} \vee I \qquad \frac{\psi}{\varphi \vee \psi} \vee I$$

The elimination rule for $\vee$ is given on the next page; it corresponds to a **proof by cases**.

Rules for $\vee$ (2)

$\vee$ elimination:

If:

- $\varphi \vee \psi$ has been derived,
- assuming φ , χ can be derived, and
- assuming ψ , χ can be derived

then it is legitimate to derive χ , removing the independent assumptions that φ and ψ hold (indicated by putting φ and ψ between square brackets):

$$\frac{\varphi \vee \psi \quad \frac{[\varphi] \dots \chi}{\chi} \quad \frac{[\psi] \dots \chi}{\chi}}{\chi} \vee E$$

Rules for $\neg$

$\neg$ introduction:

If, assuming φ , a contradiction can be derived, then it is legitimate to derive $\neg\varphi$, removing the assumption that φ holds (indicated by putting φ between square brackets):

$$\frac{\begin{array}{c} [\varphi] \\ \vdots \\ \perp \end{array}}{\neg\varphi} \neg I$$

$\neg$ elimination:

If both φ and $\neg\varphi$ have been derived, then it is legitimate to derive a contradiction:

$$\frac{\varphi \quad \neg\varphi}{\perp} \neg E$$

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Rules for $\leftrightarrow$ (2)

$\leftrightarrow$ elimination:

If $\varphi \leftrightarrow \psi$ and φ , respect. ψ , have been derived, then it is legitimate to derive ψ , respect. φ :

$$\frac{\varphi \quad \varphi \leftrightarrow \psi}{\psi} \leftrightarrow E \qquad \frac{\psi \quad \varphi \leftrightarrow \psi}{\varphi} \leftrightarrow E$$

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Rules for $\leftrightarrow$ (1)

$\leftrightarrow$ introduction:

If:

- assuming φ , ψ can be derived, and
- assuming ψ , φ can be derived

then it is legitimate to derive $\varphi \leftrightarrow \psi$, removing the assumptions that φ and ψ hold (indicated by putting φ and ψ between square brackets):

$$\frac{\begin{array}{c} [\varphi] \\ \vdots \\ \psi \end{array} \quad \begin{array}{c} [\psi] \\ \vdots \\ \varphi \end{array}}{\varphi \leftrightarrow \psi} \leftrightarrow I$$

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Example 1

The following is a proof that $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$ is valid:

$$\frac{\frac{\frac{[\varphi \wedge \psi]^1}{\psi} \wedge E \quad \frac{[\varphi \wedge \psi]^1}{\varphi} \wedge E}{\psi \wedge \varphi} \wedge I}{(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)} \rightarrow I_1$$

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Example 2

The following is a proof that $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \wedge \psi) \rightarrow \chi)$ is valid:

$$\frac{\frac{\frac{[\varphi \wedge \psi]^1}{\psi} \wedge E \quad \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge E \quad [\varphi \rightarrow (\psi \rightarrow \chi)]^2}{\psi \rightarrow \chi} \rightarrow E}{\chi} \rightarrow I_1}{(\varphi \wedge \psi) \rightarrow \chi} \rightarrow I_2$$

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Example 3

The following is a proof that $\varphi \rightarrow \neg\neg\varphi$ is valid:

$$\frac{\frac{\frac{[\varphi]^2 \quad [\neg\varphi]^1}{\perp} \neg E}{\neg\neg\varphi} \neg I_1}{\varphi \rightarrow \neg\neg\varphi} \rightarrow I_2$$

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Example 4

The following is a proof that $(\varphi \vee \chi) \wedge (\psi \vee \chi)$ is a logical consequence of $(\varphi \wedge \psi) \vee \chi$:

$$\frac{\frac{(\varphi \wedge \psi) \vee \chi \quad \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge E \quad \frac{[\chi]^1}{\varphi \vee \chi} \vee I}{\varphi \vee \chi} \vee E_1 \quad \frac{(\varphi \wedge \psi) \vee \chi \quad \frac{\frac{[\varphi \wedge \psi]^2}{\psi} \wedge E \quad \frac{[\chi]^2}{\psi \vee \chi} \vee I}{\psi \vee \chi} \vee E_2}{(\varphi \vee \chi) \wedge (\psi \vee \chi)} \wedge I$$

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Example 5

The following is a proof that $\neg\varphi \vee \neg\psi$ is a logical consequence of $\neg(\varphi \wedge \psi)$:

$$\frac{\neg(\varphi \wedge \psi) \quad \frac{\frac{[\neg(\neg\varphi \vee \neg\psi)]^3 \quad \frac{[\neg\varphi]^1}{\neg\varphi \vee \neg\psi} \vee I}{\perp} \neg E \quad \frac{[\neg(\neg\varphi \vee \neg\psi)]^3 \quad \frac{[\neg\psi]^2}{\neg\varphi \vee \neg\psi} \vee I}{\perp} \neg E}{\frac{\perp}{\neg\varphi \vee \neg\psi} \text{ RAA}_3} \text{ RAA}_1 \quad \text{ RAA}_2$$

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Intuitionistic logic (1)

Intuitionistic logic drops the inference rule RAA:

$$\frac{[\neg\varphi] \dots \perp}{\varphi} \text{RAA}$$

Intuitionistic logicians disagree with the notion of truth we have defined, and accept only **constructive proofs**.

As a consequence, they reject not only proofs by contradictions, but also many rules of inference that are valid in classical logic, including:

- the **law of excluded middle**, namely, the validity of $\varphi \vee \neg\varphi$;
- **double negation elimination**, namely, the validity of $\neg\neg\varphi \rightarrow \varphi$.

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Intuitionistic logic (2)

Actually, RAA, the law of excluded middle, and double negation elimination are **equivalent**: any one of them (together with the other inference rules of Natural deduction) enables to prove the other two.

For instance, the law of excluded middle can be proved from RAA:

$$\frac{\frac{\frac{[\varphi]^1}{\varphi \vee \neg\varphi} \vee_I \quad [\neg(\varphi \vee \neg\varphi)]^2}{\perp} \neg E \quad \frac{\perp}{\neg\varphi} \neg I_1}{\varphi \vee \neg\varphi} \vee_I \quad \frac{[\neg(\varphi \vee \neg\varphi)]^2}{\varphi \vee \neg\varphi} \text{RAA}_2 \neg E$$

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