Tutorial Solution 2

COMP 2411, Session 1, 2004

Q1: The first formula is clearly true when q is true and r is false, hence is satisfiable. The second formula is clearly true when p is false, q is false and r is false, hence is also satisfiable.

Q2:

- $(p \vee q) \wedge (\neg q \vee r)$ is in CNF and logically equivalent to the DNF $p \wedge \neg q \vee p \wedge r \vee q \wedge r$.
- $\neg p \vee (q \rightarrow \neg r)$ is logically equivalent to the CNF $\neg p \vee \neg q \vee \neg r$ and to the DNF $\neg p \vee \neg q \vee \neg r$.
- $p \wedge \neg q \vee p \wedge r$ is in DNF and logically equivalent to the CNF $p \wedge (\neg q \vee p) \wedge (p \vee r) \wedge (\neg q \vee r)$
- $p \vee q \leftrightarrow \neg r$ is logically equivalent to the CNF $(\neg r \vee \neg p) \wedge (\neg r \vee \neg q) \wedge (p \vee q \vee r)$ and to the DNF $p \wedge \neg r \vee q \wedge \neg r \vee \neg p \wedge \neg q \wedge r$.

Q3: For $n = 1$ take $X = \{p\}$.

For $n = 2$ take $X = \{p_1, p_1 \rightarrow p\}$.

For $n > 2$ take $X = \{p_1, p_1 \rightarrow p_2, \ldots, p_{n-2} \rightarrow p_{n-1}, p_{n-1} \rightarrow p\}$.

Q4: The following claim is immediately verified by induction.

> Any formula of the form $\overbrace{p \leftrightarrow \ldots \leftrightarrow p}^{n}$ with $n \in \mathbb{N} \setminus \{0\}$ is valid iff n is even; if n is odd and p is false then the formula is false.

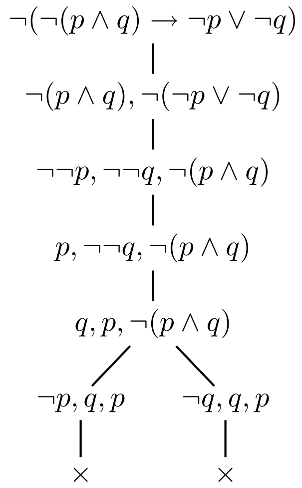
Let φ be a formula that contains $\leftrightarrow$ as its only boolean operator. Since $\leftrightarrow$ is commutative and associative, φ is logically equivalent to a formula of the form

$$(\overbrace{p_1 \leftrightarrow \ldots \leftrightarrow p_1}^{n_1}) \leftrightarrow \ldots \leftrightarrow (\overbrace{p_k \leftrightarrow \ldots \leftrightarrow p_k}^{n_k})$$

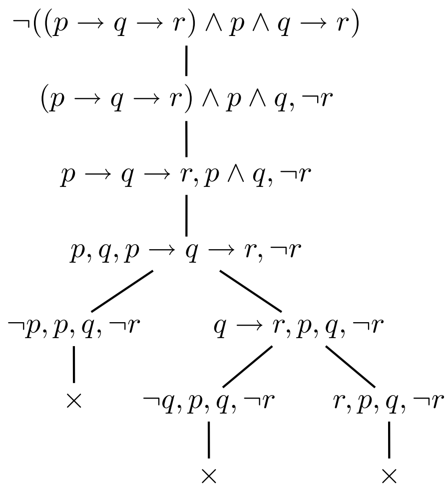
for some nonnull $k, n_1, \ldots, n_k \in \mathbb{N}$ and pairwise distinct propositional atoms $p_1 \ldots, p_k$.

- Assume that $n_1, \ldots, n_k$ are all even. By the claim above, $\overbrace{p_i \leftrightarrow \ldots \leftrightarrow p_i}^{n_i}$ is valid for all members i of $\{1, \ldots, k\}$, which clearly implies that φ is valid.
- Assume that not all of $n_1, \ldots, n_k$ are even. Without loss of generality we can suppose that n_1 is odd. Obviously, if $p_2, \ldots, p_k$ are given the value true then $\overbrace{p_i \leftrightarrow \ldots \leftrightarrow p_i}^{n_i}$ is true for all $i \in \{2, \ldots, k\}$, hence $(\overbrace{p_1 \leftrightarrow \ldots \leftrightarrow p_2}^{n_2}) \leftrightarrow \ldots \leftrightarrow (\overbrace{p_k \leftrightarrow \ldots \leftrightarrow p_k}^{n_k})$ is also true. By the claim above, $\overbrace{p_1 \leftrightarrow \ldots \leftrightarrow p_1}^{n_1}$ is false when p_1 gets the value false. Hence φ is false when p_1 gets the value false and $p_2, \ldots, p_k$ get the value true, and we conclude that φ is not valid.

Q5: We show that the negation of the formula is unsatisfiable.

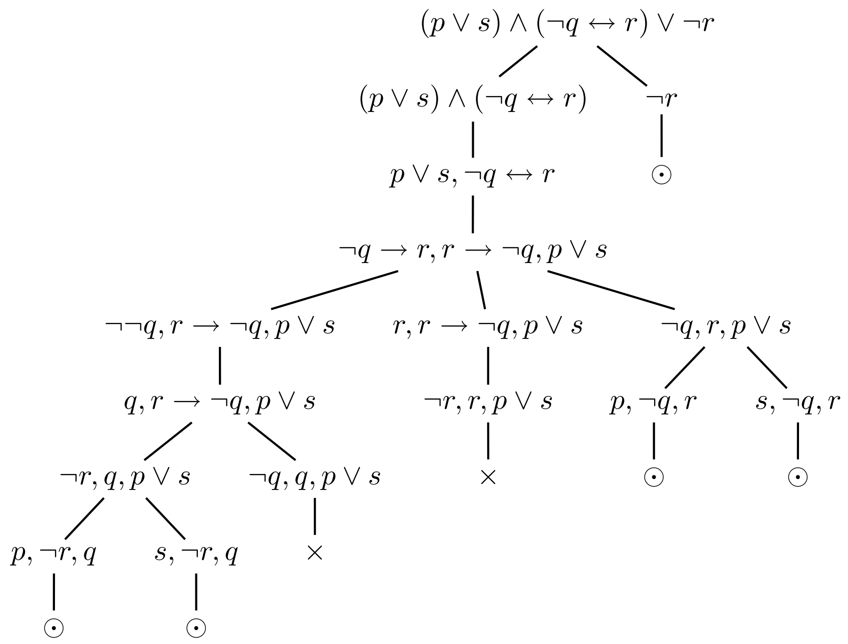


Q6: We show that $(p \to q \to r) \wedge p \wedge q \to r$ is valid, hence that the negation of the former formula is unsatisfiable.



Q7: The next tableau shows that the formula is true when:

- p is true, q is false and r is true, or
- q is false, r is true and s is true, or
- r is false



Q8: Let $n \in \mathbb{N}$ and n atomic formulas $p_1, \ldots, p_n$ be given. Consider the formula φ:

$$(l_1^0 \vee \ldots \vee l_n^0) \wedge \ldots \wedge (l_1^{2^n-1} \vee \ldots \vee l_n^{2^n-1})$$

where for all $i \in \{0, \ldots, 2^n-1\}$ and $j \in \{1, \ldots, n\}$, l_j^i is p_j if the jth bit in the binary representation of i is 1, and $\neg p_i$ otherwise. Note that φ is unsatisfiable. It is immediately verified that the number of leaves in a tableau for φ is equal to n^{2^n}.

Optimizations like applying an α-rule before a β-rule, or closing a branch even when its leaf is not labeled with literals only, do not help much. For a practical confirmation, add

```
t1a :- test1((p v q) ^ (p v ~q) ^ (~p v q) ^ (~p v ~q)).
t2a :- test2((p v q) ^ (p v ~q) ^ (~p v q) ^ (~p v ~q)).

t1b :- test1((p v q v r) ^ (p v q v ~r) ^ (p v ~q v r) ^ (p v ~q v ~r) ^ %
                  (~p v q v r) ^ (~p v q v ~r) ^ (~p v ~q v r) ^ (~p v ~q v ~r)).
t2b :- test2((p v q v r) ^ (p v q v ~r) ^ (p v ~q v r) ^ (p v ~q v ~r) ^ %
                  (~p v q v r) ^ (~p v q v ~r) ^ (~p v ~q v r) ^ (~p v ~q v ~r)).
```

to `tableau_tests.pl` and test these four queries.