Tutorial solutions 3

COMP 2411, Session 1, 2004

Q1: Proof that $\{\varphi \to \psi, \psi \to \chi\} \models \varphi \to \chi$:

1.	φ	Extra hypothesis
2.	$\varphi \to \psi$	Hypothesis
3.	ψ	MP 1,2
4.	$\psi \to \chi$	Hypothesis
5.	χ	MP 3,4
6.	$\varphi \to \chi$	Deduction rule 1,5

Proof that $\{\varphi{\to}\psi{\to}\chi, \psi\} \models \varphi{\to}\chi$:

1.	φ	Extra hypothesis
2.	ψ	Hypothesis
3.	$\varphi{\to}\psi{\to}\chi$	Hypothesis
4.	$\psi \to \chi$	MP 1,3
5.	χ	MP 2,4
6.	$\varphi \to \chi$	Deduction rule 1,5

2: Proof that $\models \neg\neg\varphi{\to}\varphi$:

1.	$(\neg\varphi{\to}\neg\neg\varphi){\to}(\neg\varphi{\to}\neg\varphi){\to}\varphi$	Axiom3
2.	$\neg\varphi{\to}\neg\varphi$	Proof in lecture notes
3.	$(\neg\varphi{\to}\neg\neg\varphi){\to}\varphi$	Q1 second part 1,2
4.	$\neg\neg\varphi{\to}\neg\varphi{\to}\neg\neg\varphi$	Axiom 1
5.	$\neg\neg\varphi{\to}\varphi$	Q1 first part 4,3

Proof that $\models \varphi{\to}\neg\neg\varphi$:

1.	$(\neg\neg\neg\varphi{\to}\neg\varphi){\to}(\neg\neg\neg\varphi{\to}\varphi){\to}\neg\neg\varphi$	axiom3
2.	$\neg\neg\neg\varphi{\to}\neg\varphi$	Previous proof
3.	$(\neg\neg\neg\varphi{\to}\varphi){\to}\neg\neg\varphi$	MP 1,2
4.	$\varphi{\to}\neg\neg\neg\varphi{\to}\varphi$	Axiom 1
5.	$\varphi{\to}\neg\neg\varphi$	Q1 first part 4,3

Proof that $\{\varphi{\to}\psi\} \models \neg\psi{\to}\neg\varphi$:

1.	$\varphi{\to}\psi$	Hypothesis
2.	$\neg\neg\varphi{\to}\varphi$	Penultimate proof
3.	$\neg\neg\varphi{\to}\psi$	Q1 first part 2,1
4.	$\psi{\to}\neg\neg\psi$	Previous proof
5.	$\neg\neg\varphi{\to}\neg\neg\psi$	Q1 first part 3,4
6.	$(\neg\neg\varphi{\to}\neg\neg\psi){\to}\neg\psi{\to}\neg\varphi$	Proof in lecture notes
7.	$\neg\psi{\to}\neg\varphi$	MP 5,6

Q3. The lecture notes give a proof that $\{\varphi, \neg\varphi\} \models \psi$. Two successive applications of the deduction

rule yield that $\varphi \to \neg\varphi \to \psi$ is valid. Since $\varphi \vee \psi$ has been defined (in this proof system) as $\neg\varphi \to \psi$, we are done.

Q4: Given a formula φ, let $\varphi^\star$ be φ with all negations in φ being deleted. It is immediately verified that for all formulas φ, if φ is an instance of axiom 1 or 2 then $\varphi^\star$ is valid. Moreover, for all formulas φ, ψ, $(\varphi \to \psi)^\star = \varphi^\star \to \psi^\star$. Hence if a formula φ_3 can be obtained by modus ponens from two formulas φ_1 and φ_2 then $\varphi_3^\star$ can be obtained by modus ponens from $\varphi_1^\star$ and $\varphi_2^\star$. We infer that if axiom scheme 3 could be proved from axioms schemes 1 and 2 with MP, then for all instances φ of axiom 3, $\varphi^\star$ would be valid. But $\varphi = (\neg p \to \neg p) \to (\neg p \to p) \to p$ is an instance of axiom 3, and $\varphi^\star = (p \to p) \to (p \to p) \to p$ is false when p is false. We conclude that axiom scheme 3 cannot be proved from axioms schemes 1 and 2 with MP.

Q5: Intuitionistic proof in Natural deduction (without RAA) that $\models (\varphi \wedge \neg\varphi) \to \psi$:

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \neg\varphi]^1}{\varphi}\,\wedge E \quad \dfrac{[\varphi \wedge \neg\varphi]^1}{\neg\varphi}\,\wedge E}{\dfrac{\bot}{\psi}\,\bot}\,\neg E}{(\varphi \wedge \neg\varphi) \to \psi}\,\to I_1$$

Q6: Intuitionistic proof in Natural deduction (without RAA) that $\{\neg(\varphi \vee \psi)\} \models \neg(\neg\varphi \to \psi)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^1}{\varphi \vee \psi}\,\vee I \quad \neg(\varphi \vee \psi)}{\dfrac{\bot}{\neg\varphi}\,\neg I_1} \quad [\neg\varphi \to \psi]^2}{\psi}\,\to E}{\varphi \vee \psi}\,\vee I \qquad \neg(\varphi \vee \psi)}{\dfrac{\bot}{\neg(\neg\varphi \to \psi)}\,\neg I_2}\,\neg E$$

Q7: Proof in Natural deduction that $\{\neg\neg\varphi\} \models \varphi$:

$$\dfrac{\dfrac{[\neg\varphi]^1 \quad \neg\neg\varphi}{\bot}\,\neg E}{\varphi}\,\mathrm{RAA}_1$$

Q8: Proof in Natural deduction that $\{\neg\varphi \to \neg\psi\} \models \psi \to \varphi$:

$$\dfrac{\dfrac{[\varphi]^2 \quad \dfrac{[\neg\psi]^1 \quad \neg\psi \to \neg\varphi}{\neg\varphi}\,\to E}{\bot}\,\neg E}{\psi}\,\mathrm{RAA}_1}{\varphi \to \psi}\,\to I_2$$