Tutorial Solutions 6

COMP 2411, Session 1, 2004

Q1: For a model $\mathfrak{M}$ of $\varphi \wedge \psi$, put $p^{\mathfrak{M}} = \{(m,n) \in \mathbb{N}^2 \mid m \geq n\}$.

For a model $\mathfrak{M}$ of $\varphi \wedge \neg\psi$, put $p^{\mathfrak{M}} = \{(m,n) \in \mathbb{N}^2 \mid m = n\}$.

For a model $\mathfrak{M}$ of $\neg\varphi \wedge \neg\psi$, put $p^{\mathfrak{M}} = \{(m,n) \in \mathbb{N}^2 \mid m > n\}$.

We show that $\neg\varphi \wedge \psi$ is not satisfiable. Let a structure $\mathfrak{M}$ be such that $\mathfrak{M} \models \neg\varphi$, hence $\mathfrak{M} \models \exists x \forall y \neg p(x,y)$. Choose $a \in |\mathfrak{M}|$ with $\mathfrak{M} \models \forall y \neg p(x,y)[\bar{a}/x]$. Suppose for a contradiction that $\mathfrak{M} \models \psi$. Choose $b \in |\mathfrak{M}|$ with $\mathfrak{M} \models \forall x p(x,y)[\bar{b}/y]$. Then $\mathfrak{M} \models p(x,y)[\bar{a}/x, \bar{b}/y]$. Hence $\mathfrak{M} \models \exists y p(x,y)[\bar{a}/x]$, which contradicts $\mathfrak{M} \models \forall y \neg p(x,y)[\bar{a}/x]$.

Q2: For instance:

φ_1: $\exists x \forall y (x \leq y)$

φ_2: $\exists y \forall x (x \leq y) \wedge \neg \exists x \forall y (x \leq y)$

φ_3: $\forall x \neg \exists y (x < y)$

φ_4: $\exists x \exists y (x < y) \wedge \neg \exists y \forall x (x \leq y)$

These structures cannot be described up to isomorphism because there is no equality in the language.

Q3: For instance:

$$\exists x \exists y \exists z (R(x,y) \wedge R(y,z) \wedge \forall u \neg R(u,x) \wedge \forall u \neg (R(x,u) \wedge R(u,y)) \wedge \forall u \neg (R(y,u) \wedge R(u,z))),$$

which implies, in particular, that 1 is smaller than 2.

Q4: Pick up a member a of the domain of $\mathfrak{M}$. Let $a_0, a_1, a_2 \ldots$ be objects that do not belong to the domain of $\mathfrak{M}$. Let $\mathfrak{N}$ be the structure whose domain is $|\mathfrak{M}| \cup \{a_i \mid i \in \mathbb{N}\}$. For all $n > 0$, n-ary predicate symbols R in V, and closed terms $t_1, \ldots, t_n$ over $|\mathfrak{M}| \cup \{a_i \mid i \in \mathbb{N}\}$, put:

$$\mathfrak{N} \models p(t_1, \ldots, t_n) \text{ iff } \mathfrak{M} \models p(t'_1, \ldots, t'_n)$$

where for all nonnull $j \leq n$, t'_j is obtained from t_j by replacing of occurrences of $\overline{a_i}$ by $\overline{a}$, for $i \in \mathbb{N}$. Intuitively, the a_i's, $i \in \mathbb{N}$, are 'copies' of a having the same properties as a with respect to the other members of the domain of $\mathfrak{M}$. It is immediately verified by induction on closed formulas that for all closed formulas φ over V, $\mathfrak{M} \models \varphi$ iff $\mathfrak{N} \models \varphi$.

When equality is available, we know that some formulas can express: "there are at most n elements in the domain", when n is a fixed nonnull natural number.

Q5. Let a structure $\mathfrak{M}$ be such that $\mathfrak{M} \models \exists x \forall y R(x,y)$. It suffices to prove that $\mathfrak{M}$ is a model of $\forall y \exists x R(x,y)$. Since $\mathfrak{M} \models \exists x \forall y R(x,y)$, we can choose $a \in |\mathfrak{M}|$ such that $\mathfrak{M} \models \forall y R(\overline{a}, y)$. Then for all $b \in |\mathfrak{M}|$, $\mathfrak{M} \models R(\overline{a}, \overline{b})$, which shows that $\mathfrak{M} \models \exists x R(\overline{a}, \overline{b})$. We conclude that $\mathfrak{M} \models \forall y \exists x R(x,y)$, as wanted.

Q6. φ is true in all structures. Indeed, let $\mathfrak{M}$ be a structure with more than n elements. Let $a_1, \ldots, a_n \in |\mathfrak{M}|$ be given. Consider the structure $\mathfrak{N}$ that is the restriction of $\mathfrak{M}$ to $\{a_1, \ldots, a_n\}$. Then $\mathfrak{N} \models \varphi$ by assumption, which implies immediately that:

$$\mathfrak{M} \models \exists y_1 \ldots \exists y_m \psi[\overline{a_1}/x_1, \ldots, \overline{a_n}/x_n].$$

We conclude that $\mathfrak{M} \models \varphi$.

Q7: Either the existential quantifier or the universal quantifier (but not both), since one quantifier is definable in terms of the other and negation. We could also remove $\rightarrow$, $\leftrightarrow$, and either disjunction or conjunction (but not both).

Q8: For instance:

```
rev_order([],[]).
rev_order([X|T],R) :- rev_order(T,R1), append(R1,[X],R).
```

The queries `rev_order([1,2,3],R)` and `rev_order(R,[1,2,3])` yield the same solution, but when asking for alternative solutions, the first query outputs `no` whereas the second one loops.

To reverse a list of length n, there are:

- $n+1$ calls to `reverse` (to reverse lists of length $n, n-1, \ldots, 1, 0$);
- the call to `reverse` on a list of length m, $m < n$, is followed by a call to `append` to append a list a length m to a list of length 1;
- to append a list a length m to some list, there are $m+1$ calls to `append`.

Hence the number of goals to solve is $n + 1 + (1 + 2 + \ldots + n) = \frac{(1+n+1)(n+1)}{2}$.

Q9. The query `top(X,Y)` yields 7 solutions:

```
X = a
Y = c

X = a
Y = d

X = b
Y = c

X = b
Y = d

X = e
Y = c

X = e
Y = d

X = e
Y = _
```

With `true(1)` replaced by cut, the 5th and 6th solutions are no longer generated.

With `true(2)` replaced by cut, the 3rd, 4th, 5th and 6th solutions are no longer generated.