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A MODEL FOR 'REAL' POKER

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A two-person poker game is analyzed and a solution is obtained. The essential difference between this game and those treated in the past is the fact that this one allows arbitrarily high bets. It turns out, curiously enough, that the number 7 is present in an essential way in both players' solutions, and furthermore, the value of the game turns out to be $\frac{1}{7}$ of the ante. I leave to the numerologists the explanation of this mystical appearance.

'REAL' poker, as it was aptly described in *Esquire*, is the game as it should be—without the sissy restrictions. Here bets, raises, and cigar smoking are completely wide open.

To date, it seems there has been no attempt to analyze a poker where there is no limit on bets. This is what we attempt in this paper. We consider a highly simplified model, but one which we feel maintains some of the spirit of 'real' poker.

The Game

A and B each ante 1 unit and are each dealt a 'hand,' namely, a randomly chosen real number in $(0,1)$. Each sees his, but not the other's, hand.

A bets any amount he chooses ($\geqq 0$).

B sees him, or folds.

The payoff is as usual.

We shall now find a solution to this game.

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IT IS INTERESTING to note that our solution is a pure strategy for both A and B in line with a conjecture of BELLMAN,[1] and some results of WEISSBLUM.[2] The value of the game is $1/7$ in favor of A.

For convenience, we denote the bet by β and let $\xi=2/(\beta+2)$.

The Solution

A bets 0 if his hand lies in $(1/7, 4/7)$; bets β with either hand $1/7\,(1-3\xi^2+2\xi^3)$ or $1-3/7\,\xi^2$

B sees the bet β if and only if his hand exceeds $1-6/7\,\xi$.

IT IS INTERESTING to look these strategies over in a *qualitative* way before we proceed to the proof.

B's strategy is quite straightforward and offers little philosophical charm. He merely sees the bet if his hand is good enough.

A's game, on the other hand, is very slick. He very systematically bluffs by betting on the hands $1/7(1-3\xi^2+2\xi^3)$ which fill the interval $(0, 1/7)$ and sometimes even suicidally, e.g., with the crummy hand 0.142815 he makes the bet of 200 units, a bet he would otherwise only make with the superb hand 0.99996 (full house, aces over fives).

A is not an absolutely wild bettor, however, and usually his high bet indicates a high hand, thus the 200 bet more usually occurs with the 0.99996 than with the 0.142815. In fact, the hands from $(4/7,1)$ are all reflected by 'honest' bets and only the smaller interval $(0,1/7)$ represents A's crookedness.

Note, however, that A bluffs $1/7$ of the time and so wins $1/7$ of the ante! The moral: Honesty is the best policy? Guess not![3]

The proof that follows is hardly an exciting one and the main point of this paper is the result and not the proof.

The Proof

We need only show that our solution is a saddle point among all pure strategies. These pure strategies are:

A bets $\beta(x)$ when dealt x.

B sees the bet β when his hand, y, lies in the set S_β. $\beta(x)$ is a certain function of x over (0,1). The S_β $(0\leqq\beta<\infty)$ are subsets of (0,1). We denote the collection of these S_β by Σ.

We first write down the expectation to A due to this pure strategy. Suppose then, that A is dealt x, B is dealt y. The gain to A is

$$\begin{array}{rll} 1 & \text{if} \quad y\notin S_{\beta(x)}, & \\ 1+\beta(x) & \text{if} \quad y\in S_{\beta(x)}, & (y<x) \\ -[1+\beta(x)] & \text{if} \quad y\in S_{\beta(x)}. & (y>x) \end{array}$$

We may write this gain as

$$1+0 \quad \text{if} \quad y \notin S_{\beta(x)},$$
$$1+\{sg(x-y)[1+\beta(x)]-1\} \quad \text{if} \quad y \in S_{\beta(x)}.$$

The total expectation, $E(\Sigma,\beta)$, then is

$$1+\iint_{y\in S_{\beta(x)}} \{sg(x-y)[1+\beta(x)]-1\}\, dx\, dy.$$

What we must prove then is the statement

$$E(\Sigma,\beta_0) \geqq E(\Sigma_0,\beta_0) \geqq E(\Sigma_0,\beta),$$

where Σ_0, β_0 are our asserted solutions.

I: Proof that $E(\Sigma,\beta_0) \geqq E(\Sigma_0,\beta_0)$

Let us introduce $S_\beta' = (a,1)$ where $a=[1-\text{measure}(S_\beta)]$; then, since the above integrand is monotone decreasing in y, we have

$$E(\Sigma,\beta) \geqq E(\Sigma',\beta).$$

Consider A's strategy (β_0). For each $\beta>0$ there are precisely two hands, x_1, x_2 that correspond to this bet. Let $S_\beta^* = S_\beta' \cap (x_1,1)+(x_2,1)$, $S_0^*=(0,1)$. Then since we have added a region to the integral wherein the integrand is $\leqq 0$ and removed a region wherein it is $\geqq 0$, we certainly have $E(\Sigma',\beta_0) \geqq E(\Sigma^*,\beta_0)$.

But now the S_β^* are intervals of the form $[\mu(\beta),1]$, where

1. $x_1 \leqq \mu(\beta) \leqq x_2 \quad \text{for} \quad \beta>0,$

2. $\mu(0)=0,$

and
$$E(\Sigma^*,\beta_0) = \int_0^1 \mu+(\beta+1)\{|\mu-x|+x-1\}\, dx.$$

But for μ satisfying **1** and **2**, we find in fact that E is *independent* of μ. This may be verified directly by splitting the above integral into $\int_0^{1/7}+\int_{1/7}^{4/7}+\int_{4/7}^{1}$, changing the independent variable to ξ, and noting that μ simply cancels out of the expression.

In particular, $E(\Sigma^*,\beta_0)=E(\Sigma_0,\beta_0)$ and ***I*** is verified.

II: Proof that $E(\Sigma_0,\beta_0) \geqq E(\Sigma_0,\beta)$

We are faced with the task of proving that when $\Sigma=\Sigma_0$, E, our expectancy integral, is maximized by β_0, that is, by the choice

1. $x=\frac{1}{7}(1-3\xi^2+2\xi^3) \quad \text{for} \quad x<\frac{1}{7},$

2. $\xi=1 \quad (\beta=0) \quad \text{for} \quad \frac{1}{7}\leqq x \leqq \frac{4}{7},$

3. $x=1-\frac{3}{7}\xi^2 \quad \text{for} \quad x>\frac{4}{7}$

E, however, is equal to

$$\int_0^1 1-\tfrac{6}{7}\,\xi+\frac{2-\xi}{\xi}\,[|1-\tfrac{6}{7}\xi-x|+x-1]\;dx.$$

Since ξ is at our disposal as a function of x we maximize E by maximizing the integrand for each value of x.

1. If $x<\frac{1}{7}$, the integrand is $-\frac{5}{7}$ independent of ξ and so we do not lose anything by choosing ξ from $x=\frac{1}{7}(1-3\xi^2+2\xi^3)$.

2. If $\frac{1}{7}\leqq x\leqq\frac{4}{7}$, the integrand has a nonvanishing derivative

$$[d|x|/dx=sg(x)],$$

and hence the maximum occurs at an endpoint. At $\xi=0$ the value (limit) is $-\frac{5}{7}$ and at $\xi=1$ the value is $2x-1$. Since $2x-1\geqq-\frac{5}{7}$, the proper choice is $\xi=1$.

3. If $\frac{4}{7}<x$, the derivative does vanish (when $x=1-\frac{3}{7}\,\xi^2$), and so this ξ as well as 0 and 1 must be considered. At 0 the integrand is $-\frac{5}{7}$; at 1 the integrand is $2x-1$; when $x=1-\frac{3}{7}\,\xi^2$ the integrand is $\frac{1}{7}(19-24\xi+6\xi^2)$. The value 0 is eliminated since, again, $2x-1\geqq-\frac{5}{7}$. Also

$$\tfrac{1}{7}(19-24\xi+6\xi^2)=1-\tfrac{6}{7}\xi^2+\tfrac{12}{7}(1-\xi)^2\geqq1-\tfrac{6}{7}\xi^2=2x-1$$

and this eliminates 1. It follows that $x=1-\frac{3}{7}\xi^2$.

Steps **1**, **2**, and **3**, taken together, constitute the required verification. It can be seen, by direct integration, that the value to A is $\mathrm{E}(\Sigma_0,\beta_0)=\frac{1}{7}$.

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2. W. Weissblum, verbal communication.
3. K. Weil and B. Brecht, *Dreigroschenoper*, Solomonns Lied.